MATH 1010E Week 8 (Martin Li)

Last time ... Taylor's Theorem.

$$f(x) = \underbrace{TP_n(x)}_{=\sum_{m=0}^{n} a_m (x-x_0)^m} + E_n(x)$$

"error term"

$x \approx x_0$ $\quad f(x) \approx TP_n(x)$

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\underbrace{(x-x_0)^{n+1}}_{\text{small}}$$

small when $x \approx x_0$

applications:

① prove inequalities: e.g. $e^x > 1 + x + \frac{x^2}{2} \quad \forall x > 0$.

② 2^{nd} derivative test: $f'(x_0) = 0,\ f''(x_0) > 0 \Rightarrow x_0$ local min.

$$f(x) \approx f(x_0) + \underbrace{f'(x_0)(x-x_0)}_{=0} + \underbrace{\frac{f''(x_0)}{2}(x-x_0)^2}_{\geq 0}$$

Recall: (Cauchy's Mean Value Thm.)

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}} \quad \text{for some } \xi \in (a,b)$$

assume $f, g : [a,b] \to \mathbb{R}$ cts on $[a,b]$, diff. in (a,b).

$g'(x) \neq 0$ for any $x \in (a,b)$ [Q: $\Rightarrow g(b) \neq g(a)$]

(Rolle)

Application (L'Hospital's Rule: how to evaluate limits of the "indeterminate form" $= \frac{0}{0}$)

E.g.

$$1 = \lim_{x \to 0} \frac{\sin x}{x} \overset{?}{=} \lim_{x \to 0} \frac{(\sin x)'}{(x)'} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

"$\frac{0}{0}$" when $x = 0$

✻

Theorem (L'Hospital's Rule) ($\lim_{x \to x_0} \frac{f(x)}{g(x)}$

Assume (i) $f, g : (a,b) \longrightarrow \mathbb{R}$ diff. in (a,b). Fix $x_0 \in (a,b)$

(ii) $f(x_0) = 0 = g(x_0)$ ($\Rightarrow$ "$\frac{0}{0}$" indeterminate form)

(iii) ~~$\lim_{x \to x_0} \frac{f'(x_0)}{g'(x_0)}$~~ $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$ exists & is finite.

Then, $\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L}$ "exists & $= L$"

Examples

(1) $$\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{2 \cancel{\sin x} \cos x}{\cancel{\sin x}} = \lim_{x \to 0} (2 \cdot \cos x) = 2 \quad \#$$

L'Hospital

[Ex: Do this by $\sin^2 x = 1 - \cos^2 x$]

(2) $$\lim_{x \to 0} \frac{x - \sin x}{x^3} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 0} \frac{\sin x}{6x} = \frac{1}{6} \quad \#$$

Note: Apply L'Hospital's rule a few times until you can evaluate the limit.

(3) $$\lim_{x \to 0} \frac{e^x - x - 1}{1 - \cosh x} = \lim_{x \to 0} \frac{e^x - 1}{-\sinh x} = \lim_{x \to 0} \frac{e^x}{-\cosh x} = \frac{1}{-1} = -1 \quad \#$$

"Proof" of L'Hospital's Rule "$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$".

$$\frac{f(x)}{g(x)} \underset{\because f(x_0)=0=g(x_0)}{=} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} \underset{\text{Cauchy's MVT}}{=} \frac{f'(\xi)}{g'(\xi)}$$ for some ξ between x_0 & x.

or ($\xi \in (x_0, x)$ or $\xi \in (x, x_0)$)

lim as $x \to x_0 \Rightarrow \xi \to x_0$

$$\Rightarrow \quad \lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{\xi\to x_0}\frac{f'(\xi)}{g'(\xi)} = L.$$

■

Remark: "Other indeterminate forms"

$$\left(\frac{0}{0} \text{ or } 0\cdot\infty \text{ or } \frac{\infty}{\infty} \text{ or } 0^0 \right)$$

✓

$(0.01)^{\frac{1}{2}} = 0.1$

Q: $0^\infty := e^{\infty \ln 0} = e^{-\infty} = 0$

not indeterminate

$\infty^\infty = \infty$

Note: L'Hospital's rule handles all of these

e.g. $\left[0\cdot\infty = 0\cdot\frac{1}{0} = \text{"}\frac{0}{0}\text{"} \right]$ AND. $\left[\frac{\infty}{\infty} = \frac{1/0}{1/0} = \text{"}\frac{0}{0}\text{"} \right]$ ....

E.g. $$\lim_{x\to 0^+} \underbrace{x\ln x}_{0\cdot(-\infty)} = \lim_{x\to 0^+}\frac{\ln x}{\underbrace{1/x}_{\frac{-\infty}{+\infty}}} \underset{\text{true}}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

win

$$\boxed{\lim_{x\to 0} x\ln x = 0}$$ ← x goes to 0 faster than $\ln x \to -\infty$

Remark: If we replace the condition " $f(x_0) = 0 = g(x_0)$ "

by " $f(x_0) = \pm\infty = g(x_0)$ " precisely, $\lim_{x\to x_0} f(x) = \pm\infty$
$= \lim_{x\to x_0} g(x)$.

then L'Hospital's rule still holds.

E.g. $\lim_{x\to\infty} \dfrac{x^2}{e^x} = \lim_{x\to\infty} \dfrac{2x}{e^x} = \lim_{x\to\infty} \dfrac{2}{e^x} = 0$

$\boxed{\begin{array}{l} \infty = +\infty \\ -\infty = -\infty \end{array}}$ " $\dfrac{\infty}{\infty}$ " " $\dfrac{\infty}{\infty}$ "

Alternatively: $\lim_{x\to\infty} \dfrac{x^2}{e^x} = \lim_{x\to\infty} \dfrac{\overset{e^{-x}}{\overset{\shortparallel}{1/e^x}}}{1/x^2} = \lim_{x\to\infty} \dfrac{-e^{-x}}{-2/x^3}$ (fails)

" $\dfrac{0}{0}$ "

E.g. $\boxed{\lim_{x\to\infty} \dfrac{x^k}{e^x} = 0}$ for any $k \geq 1$.

↳ e^x goes to ∞ faster than any $x^k \to \infty$

as $x \to +\infty$.

Tricky Examples:

(A) $\lim_{x\to\infty} x \sin\dfrac{1}{x} \overset{u = 1/x}{\underset{\phi}{=}} \lim_{u\to 0} \dfrac{1}{u} \sin u = \lim_{u\to 0} \dfrac{\sin u}{u} = 1$ #.

" $\infty \cdot 0$ "

$= \lim_{x\to\infty} \dfrac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \dfrac{(\cos\frac{1}{x})\cancel{(-\frac{1}{x^2})}}{\cancel{(-\frac{1}{x^2})}} = 1$

① $\dfrac{\sin 1/x}{1/x} = \dfrac{0}{0}$

② $\dfrac{x}{1/\sin\frac{1}{x}} = \dfrac{\infty}{\infty}$

$= \lim_{x\to\infty} \dfrac{x}{\frac{1}{\sin\frac{1}{x}}} = \lim_{x\to\infty} \dfrac{1}{\left(-\frac{1}{\sin^2\frac{1}{x}}\right)\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)} = ?$

(B) $\underline{\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)} = \lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) = \text{"}\frac{0}{0}\text{"}$

"$\infty - \infty$"

$$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x + x\cos x}\right) = \text{"}\frac{0}{0}\text{"}$$

[Ex: $\infty + \infty = +\infty$

$-\infty - \infty = -\infty$]

$$= \lim_{x\to 0}\frac{\sin x}{\cos x + (\cos x - x\sin x)}$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2}$$

$$= 0 \quad \#.$$

"Bad" Approach: $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x}\right) = \lim_{x\to 0} 0 = 0.$

$\sin x \sim x$

as $x\to 0$

(C) $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x}$

∞^0

$$\overset{?}{=} e^{\lim_{x\to\infty}\left(\frac{1}{x}\ln x\right)}$$

$$= e^{\lim_{x\to\infty}\frac{\ln x}{x}} = \frac{\infty}{\infty}$$

$$= e^{\lim_{x\to\infty}\frac{1/x}{1}} = e^0 = 1 \quad \#$$

f continuous at x_0.

"$\lim_{x\to x_0} f(x) = f(x_0)$

$= f\left(\lim_{x\to x_0} x\right)$"

"Proof" of L'Hospital's Rule for $\frac{\infty}{\infty}$.

① $f, g : (a,b) \to \mathbb{R}$ diff., $x_0 \in (a,b)$

② "$f(x_0) = \pm\infty = g(x_0)$"

③ $\displaystyle\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L$.

$\Rightarrow \displaystyle\lim_{x\to x_0} \frac{f(x)}{g(x)} = L$

$$\frac{\infty}{\infty} = \frac{1/0}{1/0} = \text{"}\frac{0}{0}\text{"}$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{1/g(x)}{1/f(x)} = \lim_{x\to x_0} \frac{-g'(x)/g(x)^2}{-f'(x)/f(x)^2}$$

"$\frac{0}{0}$"

$$= \lim_{x\to x_0} \left(\frac{g'(x)}{f'(x)} \cdot \frac{f(x)^2}{g(x)^2} \right)$$

💣
$$= \lim_{x\to x_0} \left(\frac{g'(x)}{f'(x)} \right) \cdot \left(\lim_{x\to x_0} \frac{f(x)}{g(x)} \right)^2$$

[Ex: think about these 💣]

Cancel out one of 💣

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{1}{\displaystyle\lim_{x\to x_0} \frac{g'(x)}{f'(x)}} \overset{💣}{=} \lim_{x\to x_0} \frac{f(x)}{g'(x)}$$

#.

Otherwise! $1 = \lim_{x\to 0} 1 = \lim_{x\to 0}\left(x \cdot \frac{1}{x}\right) = \left(\lim_{x\to 0} x\right) \cdot \left(\lim_{x\to 0} \frac{1}{x}\right) = 0 \cdot (-) = 0$

Recall: $\lim_{x\to 0} x \sin\frac{1}{x} \neq \left(\lim_{x\to 0} x\right) \cdot \left(\lim_{x\to 0} \sin\frac{1}{x}\right) = 0 \cdot (\sim) = 0$

Recall: MATH 1010 Calculus. (1D)

① limits

② Differentiation $f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ } done!

③ Integration (indefinite or definite)

$$\begin{tikzcd} f(x) \arrow[r, bend left, "\frac{d}{dx} \text{ differentiate}"] & f'(x) \arrow[l, bend left, "\int dx \text{ integration}"] \end{tikzcd}$$

④ Fundamental Theorem of Calculus

Indefinite integral

Q: Given a function $f: (a,b) \longrightarrow \mathbb{R}$ (cts / diff.)

can we find another function $F: (a,b) \longrightarrow \mathbb{R}$ s.t.

primitive

$$\underset{\text{unknown}}{F'(x)} = \underset{\text{given}}{f(x)} \qquad \forall x \in (a,b)$$

Simple examples:

① Let $f(x) = e^x$.

Solve for F s.t. $\boxed{F'(x) = e^x}$

Known: $F(x) = e^x$ is a solution

Q: any other solutions? Yes! e.g. $F(x) = e^x + 1$

$$F(x) = e^x + \underset{\text{constant}}{C}$$

Q: are these ALL the solutions?

Yes! [Ex: related to " $f'(x)=0 \ \forall x \in (a,b)$

$\Rightarrow f(x) = c$ "]

constant

Def$^{\underline{n}}$: If $F'(x) = f(x)$, then we say that $F(x)$ is the ~~inte~~ "indefinite integral" of $f(x)$

$$\int f(x)\,dx := F(x) + C$$

integration constant.

Eg. $\int e^x\,dx = e^x + C$ ※

Eg.: $\int 1\,dx = x + C$ ※

Proposition:

(1) $\int \cos x\,dx = \sin x + C$

(2) $\int \sin x\,dx = -\cos x + C$

(3) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ for any $n = 0,1,2,3,\ldots$

[Remark: True for $n \in \mathbb{R}$ since $\frac{d}{dx} x^{\alpha} = \alpha x^{\alpha-1}$
except for $n=-1$]

(4) $\int \frac{1}{x}\,dx = \ln|x| + C$ [Note: $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for any $x \neq 0$]

$x<0$, $\frac{d}{dx}\ln(-x) = \left(\frac{1}{-x}\right)(-1) = \frac{1}{x}$.

Property I: (linearity)

(1) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

(2) $\int k\cdot f(x)\,dx = k\int f(x)\,dx$ $\quad k = \text{constant.}$

E.g. (a) $\int (x^4 - 3x + 7)\,dx$

$$= \int x^4\,dx - 3\int x\,dx + 7\int 1\,dx$$

$$= \left(\frac{x^5}{5} + C_1\right) - 3\left(\frac{x^2}{2} + C_2\right) + 7\left(x + C_3\right)$$

$$= \underbrace{\frac{x^5}{5} - \frac{3x^2}{2} + 7x}_{F(x).} + \underbrace{C_1 - 3C_2 + 7C_3}_{=C}$$

(b) $\int \frac{(x+2)^2}{x}\,dx$ $\quad$ [Q: $\int \frac{(x+2)^2}{x+1}\,dx = ?$]

$$= \int \frac{x^2+4x+4}{x}\,dx$$

$$= \int \left(x + 4 + \frac{4}{x}\right)dx$$

$$= \int x\,dx + \int 4\,dx + 4\int \frac{1}{x}\,dx$$

$$= \frac{x^2}{2} + 4x + 4\ln|x| + C$$

#

(c) $\displaystyle\int \frac{5x^2+\sqrt{x}+3}{\sqrt{x}}\,dx$

$$= \int \left(5x^{\frac{3}{2}} + 1 + 3x^{-\frac{1}{2}}\right) dx$$

$$= 5\,\frac{x^{\frac{5}{2}}}{5/2} + x + 3\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2x^{\frac{5}{2}} + x + 6x^{\frac{1}{2}} + C \quad \#.$$

Examples:

(1) $\displaystyle\int \sec^2 x\,dx = \tan x + C$

(2) $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

(3) $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

(4) $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

(5) $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

Pf. $\displaystyle\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$ mystenous !!

why this one ?

$= \cdots\cdots$ (Ex:)

(6) $\displaystyle\int \csc x\,dx = ?$ (Ex:)